Spectra of algebraic structures

Alberto Facchini
Università di Padova, Italy

Lens, 6 July 2021

This talk is dedicated to Syed Tariq Rizvi.

We are all tired of online talks and online teaching.

Which is the worst?

Which is the worst?

zoom

Which is the worst?

zoom,

Microsoft Teams

Which is the worst?

zoom,

Microsoft Teams, or

BigBlueButton?

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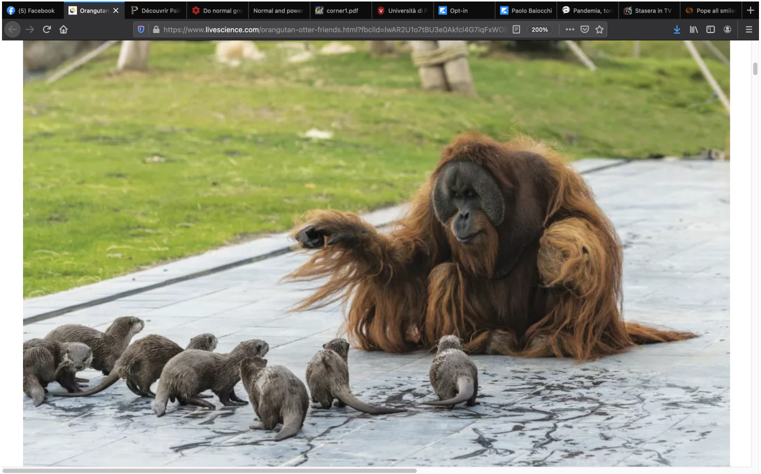
The picture is from a zoo in Belgium: Pairi Daiza, cf. https://www.livescience.com/orangutan-otter-friends.html

Face to face teaching



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Now let's begin with the serious part of talk.

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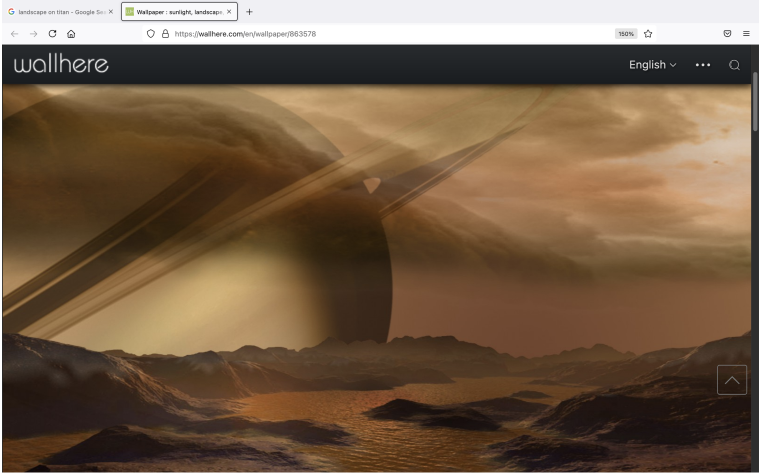
Commutative rings $\Rightarrow$ Topology is not necessary! $\Rightarrow$ Spectra of rings.

Every time, like landing on a new planet



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Commutative monoids, abelian ℓ-groups, prime spectrum of an MV-algebra, Hofmann-Lawson spectrum of a continuous lattice, Zariski-Riemann spaces,. . .

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Why are spectral spaces so frequent in nature? Any deep reason? Explanation?

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A wonderful paper

M. Reyes, Obstructing extensions of the functor Spec to noncommutative rings, Israel J. Math. 192 (2012), 667–698.

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Then extended in

M. Ben-Zvi, A. Ma and M. Reyes, A Kochen-Specker theorem for integer matrices and noncommutative spectrum functors, J. Algebra 491 (2017), 280–313.

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Theorem

Let F be a contravariant functor from the category of rings to the category Top *whose restriction to the full subcategory of commutative rings is naturally isomorphic to the functor* Spec*. Then F assigns the empty topological space to the rings of matrices $\mathbb{M}_n(R)$ for any ring R and any integer $n \geq 3$.*

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Corollary

Let $G\colon \mathsf{Rings} \to \mathsf{CommRings}$ *be a covariant functor from the category* Rings *of rings to the category* CommRings *of commutative rings whose restriction to the full subcategory of commutative rings is naturally isomorphic to the identity functor* $\mathsf{CommRings} \to \mathsf{CommRings}$*. Then G assigns the zero ring to the rings of matrices* $\mathbb{M}_n(R)$ *for any ring R and any integer $n \geq 3$.*

The correct setting: lattices

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In all the previous examples, there is a multiplicative lattice around:
For commutative rings: the lattice of its ideal with multiplication of ideals.
For noncommutative rings: the lattice of its two-sided ideal with multiplication of ideals, or $IJ + JI$ as a product, if you prefer.
For groups: the modular lattice of its normal subgroups with commutator of two normal subgroups.
For lattices: the lattice itself with multiplication $xy := x \wedge y$.

Multiplicative lattices

Multiplicative lattice are an algebraic structure to which little attention has been devoted, but which already appear in Krull (1924!), and has been studied by M. Ward (1937), Ward and R.P. Dilworth (1937), D.D. Anderson (1974), E.W. Johnson, and J.A. Johnson (1970), Hofmann and Keimel (1978), quantales, frames, locales, . . .

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In all these papers, further axioms are required: associativity or commutativity of multiplication, distributivity with $\vee$, identity, compatibility of multiplication and partial order, the multiplication is the meet, . . .

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An element $p \neq 1$ is said to be *prime* if it satisfies the implication

$$xy \leq p \Rightarrow (x \leq p \text{ or } y \leq p).$$

Let $\mathrm{Spec}(L)$ be the set of all prime elements of L.

The mapping V

We have a mapping

$$\begin{aligned} V &\colon L \to \mathcal{P}(\mathrm{Spec}(L)) \\ V &\colon x \mapsto V(x) := \{\, p \in \mathrm{Spec}(L) \mid x \leq p \,\}. \end{aligned}$$

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The mapping $V: L \to \mathcal{P}(\mathrm{Spec}(L))$ has the following properties:

The mapping V

(1) V transforms the multiplication in L into the union in $\mathcal{P}(\mathrm{Spec}(L))$, that is, V is a magma morphism of the magma $(L, \cdot)$ into the magma (the commutative monoid) $(\mathcal{P}(\mathrm{Spec}(L)), \cup)$:

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(2) V transforms the $\vee$ in L into the intersection in $\mathcal{P}(\mathrm{Spec}(L))$ (more is true: it transforms an arbitrary $\bigvee$ in L into an arbitrary intersection in $\mathcal{P}(\mathrm{Spec}(L))$, even in the infinite case):

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(3) The image $V(L)$ of the mapping V satisfies the axioms for the closed sets of a topology on $\mathrm{Spec}(L)$.

$\mathrm{Spec}(L)$ with this topology is called the *Zariski spectrum* of L.

Always a sober space

Lemma

$\mathrm{Spec}(L)$ *is a sober space.*

Irreducible topological spaces

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In the complete lattice of closed subsets of a topological space X, irreducible closed subsets are exactly the join-irreducible elements of the lattice.

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For example, $\mathrm{Spec}(R)$ is sober: its irreducible closed subsets are the subsets $V(\mathfrak{p})$ with $\mathfrak{p}$ a prime ideal of R, that is, the closures $V(\mathfrak{p})$ of the points $\mathfrak{p}$ of $\mathrm{Spec}(R)$.

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It is that, in some sense, sober spaces are (bounded distributive) lattices, in the sense that the lattice $\Omega(X)$ of open subsets of a sober space X completely determines the underlying set X.

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The category of complete join-semilattice is a nice symmetric monoidal closed category.

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The category MCL of multiplicative lattices

In the category MCL, whose objects are our complete multiplicative lattices $(L, \cdot)$, morphisms $L \to M$ are monotone Galois connections $(f, u) \colon L \to M$ such that $f(x)f(x') \leq f(xx')$ for every $x, x' \in L$. (Hence, morphisms in MCL = morphisms in the category of complete join-semilattices such that $f(x)f(x') \leq f(xx')$ for every $x, x' \in L$.)

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Composition in MCL is defined by $(f', u') \circ (f, u) = (f' \circ f, u \circ u')$ for every pair of morphisms $(f, u)\colon L \to L'$ and $(f', u')\colon L' \to L''$.

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Clearly, the composite functor of the two functors

$$\mathsf{CommRings} \to \mathsf{MCL} \quad \text{and} \quad \mathrm{Spec}\colon \mathsf{MCL} \to \mathsf{Top}$$

is the usual contravariant functor Spec from the category of commutative rings with identity to the category Top of topological spaces.

Spec is a right adjoint

The functor $\mathrm{Spec}\colon \mathsf{MCL}^{\mathrm{op}} \to \{\,\text{sober spaces}\,\}$ is a right adjoint of the functor $\{\,\text{sober spaces}\,\} \to \mathsf{MCL}^{\mathrm{op}}$, that maps any sober space X to the complete lattice $\Omega(X)$ of its open subsets, with multiplication the intersection: $xy = x \wedge y$ for every $x, y \in \Omega(X)$.

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"I don't like those multiplicative lattices. Arent' lattices sufficient? You just call an element p of a lattice L *prime* if $x \wedge y \leq p \Rightarrow (x \leq p \text{ or } y \leq p)$." Ok, by this doesn't even cover the first original examples of commutative rings with identity. If you take a DVR, with this definition of prime, all proper ideals of R would be prime, not only 0 and the maximal ideal of R as we want!

Spectrum of a bounded distributive lattice

Let D be a bounded distributive lattice, and let $X := \mathrm{Spec}(D)$ denote the set of prime ideals of D.

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Its inverse is

$K^\circ(-)\colon \{\text{topol. spectral spaces}\} \to \{\text{bounded distributive lattices}\},$

$$X \mapsto K^\circ(X).$$

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Output: a bounded distributive lattice. Their category is equivalent to the category of spectral spaces.